

Multiple integrals: double, triple, polar, cylindrical, spherical

Key-points printable for <https://lessons.ihsan.cc/ece206/multiple-integrals/> – key formulas, worked examples, and Problem Set 2 with answers.

1 Double integrals

Lecture 4, pp. 1–3.

$$\iint_D f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i) \Delta A_i.$$

With $f = 1$ this gives the area of D ; with f a surface density it gives the total mass or charge.

On a rectangle ($x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$), the order doesn't matter, and a product integrand splits:

ON THE FORMULA SHEET

$$\iint_D f dA = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f dx dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f dy dx, \quad \iint_D g(x)h(y) dA = \int_{y_1}^{y_2} h dy \cdot \int_{x_1}^{x_2} g dx.$$

2 Regions and order

Lecture 4, pp. 4–7.

Two descriptions of a plane region fix the order of integration – the variable with function limits is integrated first:

$$(1) \ x_\ell \leq x \leq x_u, \ y_\ell(x) \leq y \leq y_u(x) \qquad (2) \ y_\ell \leq y \leq y_u, \ x_\ell(y) \leq x \leq x_u(y).$$

To switch order, redraw the region from the limits you have and describe it the other way – never swap limits in place; choose the root of a solved boundary from where the region lies.

WORKED EXAMPLE --

FORCING A SWITCH

(LECTURE 4 P.7)

$\int_0^1 \int_x^1 e^{-y^2} dy dx$: e^{-y^2} has no elementary antiderivative, so switch. The triangle $0 \leq x \leq 1$, $x \leq y \leq 1$ is also $0 \leq y \leq 1$, $0 \leq x \leq y$.

$$\int_0^1 \int_0^y e^{-y^2} dx dy = \int_0^1 y e^{-y^2} dy = \left[-\frac{1}{2} e^{-y^2} \right]_0^1 = \frac{1 - e^{-1}}{2} \approx 0.316.$$

PROBLEM SET 2 -- Q1 · CORE · OFFICIAL

(a) $\iint_D (x+y) dA$, triangle $(0,0), (0,1), (2,1)$. (b) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3} dx dy$. (c) $\iint_D (x^2+y^2) dA$, D bounded by $y = x^2$, $x = y^2$. (d) $\int_{-2}^0 \int_{-2}^x \frac{x}{\sqrt{x^2+y^2}} dy dx$.

Answers: (a) $\frac{4}{3}$ (b) $\frac{2}{9}(2\sqrt{2}-1)$ (c) $\frac{6}{35}$ (d) $2(1-\sqrt{2})$.

3 Change of variables

Lecture 4, pp. 8-12.

The Jacobian is the local area scale factor of a map $(u,v) = F(x,y)$; the Jacobian of the inverse map is its reciprocal:

ON THE FORMULA SHEET

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}.$$

Theorem 1. For $(x,y) = F(u,v) = (f(u,v), g(u,v))$ mapping D_{uv} one-to-one onto D_{xy} , with G continuous on D_{xy} :

ON THE FORMULA SHEET

$$\iint_{D_{xy}} G(x,y) dx dy = \iint_{D_{uv}} G(f(u,v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv.$$

Pick u, v so each piece of the boundary is a constant, which makes D_{uv} a rectangle.

WORKED EXAMPLE --
THE FOUR-CURVES
REGION (LECTURE 4
P.11)

D bounded by $y = 1/x$, $y = 2/x$, $y = x^2$, $y = 2x^2$: let $u = xy$, $v = y/x^2$, giving $1 \leq u \leq 2$, $1 \leq v \leq 2$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{3y}{x^2} = 3v \implies \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{3v}.$$

$$A(D_{xy}) = \frac{1}{3} \int_1^2 \int_1^2 \frac{1}{v} du dv = \frac{1}{3} \ln 2.$$

PROBLEM SET 2 -- Q5 · CORE · OFFICIAL

$(u, v) = F(x, y) = (y/x^2, x/y^2)$, $x, y > 0$. (a) Find $\partial(x, y)/\partial(u, v)$. (b) Area of $D_{xy} = \{x^2 \leq y \leq 2x^2, 2y^2 \leq x \leq 3y^2, x, y > 0\}$.

Answers: (a) $\frac{1}{3u^2v^2}$ (b) $\frac{1}{36}$.

4 Polar coordinates

Lecture 4, p. 13.

Polar is Theorem 1 with $(x, y) = (r \cos \theta, r \sin \theta)$, Jacobian r :

ON THE FORMULA SHEET

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \quad \iint_{D_{xy}} G(x, y) dx dy = \iint_{D_{r\theta}} G(r \cos \theta, r \sin \theta) r dr d\theta.$$

Use polar when the boundary is circles about the origin and rays from it, or the integrand is built from $x^2 + y^2$.

WORKED EXAMPLE --
LECTURE 4 P.13

$D = \{1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq 0, y \leq x\}$ is $1 \leq r \leq 2$, $0 \leq \theta \leq \pi/4$.

$$\begin{aligned} \iint_D xy dA &= \int_0^{\pi/4} \int_1^2 r^3 \cos \theta \sin \theta dr d\theta \\ &= \left[\frac{r^4}{4} \right]_1^2 \cdot \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/4} = \frac{15}{4} \cdot \frac{1}{4} = \frac{15}{16}. \end{aligned}$$

PROBLEM SET 2 -- Q2, Q3, Q6 · INTRO core / challenge · official

Q2: volume under $z = 2 - x^2 - y^2$, above $z = 0$, inside $x^2 + y^2 = 1$. **Q3:** $\iint_D (x^2 + y^2)^{3/2} dA$, $D = \{1 \leq x^2 + y^2 \leq 4, y \geq \sqrt{3}x, y \geq x/\sqrt{3}\}$. **Q6:** $\iint_D x^2 dA$, D the ellipse $10x^2 + 6xy + y^2 = 2$ (hint: $u = x/\sqrt{2}$, $v = (3x + y)/\sqrt{2}$ maps it to the unit disk).

Answers: Q2 $\frac{3\pi}{2}$ Q3 $\frac{31\pi}{6}$ Q6 π .

5 Triple integrals

Lecture 4, pp. 14–16; Lecture 5, pp. 1–5.

$$\iiint_D f(x, y, z) dV = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i.$$

With D between $z_\ell(x, y)$ and $z_u(x, y)$ over the shadow D_{xy} :

ON THE FORMULA SHEET

$$\iiint_D f(x, y, z) dV = \iint_{D_{xy}} \int_{z_\ell(x, y)}^{z_u(x, y)} f(x, y, z) dz dA.$$

Any of x, y, z may be innermost, giving $3! = 6$ possible orders; each inner limit may depend only on the variables outside it.

WORKED EXAMPLE --
THE TETRAHEDRON
(LECTURE 5 P.4)

$\iiint_D z dV$ over the tetrahedron with vertices $(a, 0, 0), (0, b, 0), (0, 0, c), (0, 0, 0)$, bounded by the coordinate planes and $x/a + y/b + z/c = 1$.

$$0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right), \quad 0 \leq y \leq b\left(1 - \frac{x}{a}\right), \quad 0 \leq x \leq a.$$

Substituting $x = as$, $y = bw$ turns the shadow into the unit triangle $s + w \leq 1$ with $dx dy = ab ds dw$:

$$\iiint_D z dV = \frac{c^2}{2} ab \int_0^1 \int_0^{1-s} (1-s-w)^2 dw ds = \frac{abc^2}{2} \cdot \frac{1}{12} = \frac{abc^2}{24}.$$

PROBLEM SET 2 -- Q7 · CORE · OFFICIAL

Volume of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$, $x + 2y + 2z - 4 = 0$. **Answer:** $\frac{8}{3}$.

6 Cylindrical and spherical coordinates

Lecture 5, pp. 6–9; primer 2, pp. 23–33.

Cylindrical keeps z and replaces x, y by polar coordinates, Jacobian r ; spherical names a point by $\rho \geq 0$ (distance from origin), ϕ (angle from $+z$ -axis) and θ (as in cylindrical), Jacobian $\rho^2 \sin \phi$:

ON THE FORMULA SHEET

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad dV = r \, dr \, d\theta \, dz.$$

ON THE FORMULA SHEET

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi, \quad dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

Use cylindrical for cylinders/paraboloids about the z -axis; spherical for spheres/cones about the z -axis. A plane becomes a variable bound in spherical coordinates: $z = c$ is $\rho = c / \cos \phi$.

WORKED EXAMPLE -- Unit ball between the cone $\phi = \pi/4$ and the xy -plane ($\phi = \pi/2$):

PRIMER 2 P.28

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{3} = \frac{\sqrt{2}\pi}{3}.$$

PROBLEM SET 2 -- Q8, Q9, Q10 · CORE *core / challenge* · official

Q8: mass of S inside $x^2 + y^2 = 1$, below $z = 4$, above $z = 1 - x^2 - y^2$, density $K\sqrt{x^2 + y^2}$. **Q9:** mass of the region between the upper half of $x^2 + y^2 + z^2 = 4$ and the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$, density $2 - z$. **Q10:** volume of the sphere $x^2 + y^2 + z^2 = 4$ cut by $z = 1$ (the part $z \geq 1$).

Answers: Q8 $\frac{12\pi K}{5}$ Q9 $\frac{7\pi}{3}$ Q10 $\frac{5\pi}{3}$.

Lesson source src/multiple_integrals.jsx, read 2026-09-25; Lectures 4 and 5, pre-lecture primer 2 (VecCalc-2) and Problem Set 2 with its official solutions, all filed under lessons/ECE206/materials/. Problem Set 2 Q4 is not covered here.