

Quiz 2, step by step

Every method on the quiz as a recipe of numbered steps, one example worked with nothing skipped, the traps that cost marks, and a self-check with its answer at the back.

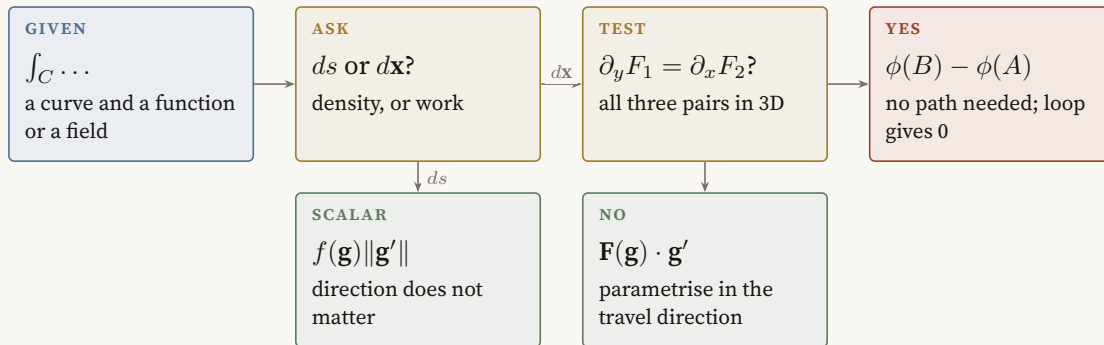
The quiz covers double and triple integrals, vector fields, line integrals of scalar and vector fields, and the fundamental theorem for line integrals. Green's theorem is not on it. Expect some true/false and multiple-choice questions with one answer each, and some written ones, at an easier level than Möbius 2.

Each section below has the same four parts. Read the recipe, then follow the worked example with a pencil, doing each line before you read it. The two places where set-ups usually stall get extra room: the norm $\|\mathbf{g}'(t)\|$ in §6, and reading limits off a region in §1 and §3. The table below is the first thing to read when a question is in front of you.

How to attack a question

IF THE QUESTION SHOWS	REACH FOR	FIRST MOVE
An iterated integral whose inner integral has no antiderivative (e^{y^2} , $\sin(x^2)$, $\sqrt{1+x^3}$)	Switch the order, §1	Sketch the region from the limits, then read it the other way.
A disk, a ring, a wedge, or $x^2 + y^2$ in the integrand	Polar, §2	$dA = r dr d\theta$; read r along a ray, θ from the bounding lines.
A solid with a cylinder, a paraboloid $z = x^2 + y^2$, or a floor and a ceiling	Cylindrical, §3	$dV = r dz dr d\theta$; z from floor to ceiling, then the shadow.
A ball, a sphere, a cone	Spherical, §3	$dV = \rho^2 \sin \phi d\rho d\phi d\theta$; ρ along a ray, ϕ down from the z -axis.
A region bounded by two pairs of curves, or a substitution handed to you	Jacobian, §4	Set u, v to the two boundary expressions; $dA = \partial(x, y)/\partial(u, v) du dv$.
“Find the field lines”, “sketch the field”	Field lines, §5	Solve $dy/dx = F_2/F_1$; arrows from $\mathbf{F}$ at a point.
$\int_C f ds$, the mass of a wire, the charge on a cord, an arc length	Scalar line integral, §6	Parametrise, then $\ \mathbf{g}'(t)\ $ in four lines.
$\int_C \mathbf{F} \cdot d\mathbf{x}$, work, circulation	Vector line integral, §7	Is the field conservative? If not, $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)$.
A messy path, “any curve from A to B ”, “path independent?”, “potential”	Conservative field, §8	Mixed partials, then ϕ , then $\phi(B) - \phi(A)$.
A closed curve and a field that passes the test (no holes)	Fundamental theorem, §8	The answer is 0. Write the test as your reason.

Two questions settle most line-integral problems before any calculus starts. Is it ds (a scalar density, no direction) or $d\mathbf{x}$ (a field doing work, direction matters)? And for $d\mathbf{x}$, does the field have a potential? Figure 1 is that decision.

FIGURE 1 *which line integral, and how*

The slate box is what you are handed. Ochre boxes are the two questions you ask, in order. Sage boxes are the computations; the rust box is the shortcut the fundamental theorem gives you.

1 Switching the order of a double integral

The limits describe a region. Draw the region, then describe the same region the other way round.

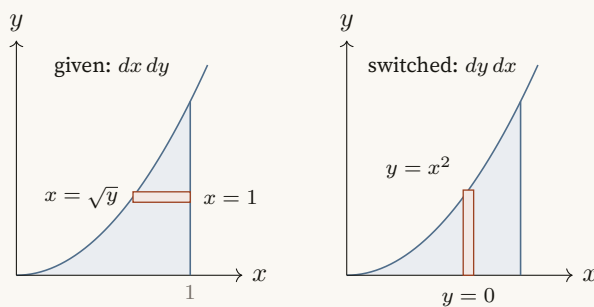
An iterated integral $\int_a^b \int_{h_1(x)}^{h_2(x)} f \, dy \, dx$ says: for each x between a and b , y runs from the lower curve h_1 to the upper curve h_2 . That sentence is a picture. Switching the order means covering the same picture with horizontal strips instead of vertical ones, so the new limits come from the picture and never from shuffling the old ones.

Recipe

1. Write the region as inequalities, straight from the limits: outer variable between two numbers, inner variable between two curves.
2. Sketch it. Draw each curve the inner limits name, and the two lines the outer limits name. Shade what lies between. Find every corner by solving the curves pairwise.
3. Draw one strip in the new direction (horizontal if the new inner variable is x). Ask where the strip enters the region and where it leaves. Those curves, solved for the new inner variable, are the new inner limits.

4. Slide the strip across the whole region. The first and last positions are the new outer limits, and they are numbers.
5. If the strip enters or leaves through a different curve partway across, split the region at that point into two integrals.

FIGURE 2 one region, two ways to cut it



The slate region is $0 \leq x \leq 1$, $0 \leq y \leq x^2$. Left: the given order cuts it into horizontal strips from $x = \sqrt{y}$ to $x = 1$. Right: after the switch, vertical strips from $y = 0$ up to $y = x^2$. The rust strip is the one you draw to read the limits.

WORKED EXAMPLE 1 · THE CLASSIC: ONLY DOABLE AFTER THE SWITCH · PS2 Q1(B)

Evaluate $I = \int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3} \, dx \, dy$.

WORKED IN FULL · NO STEP SKIPPED

Why switch. The inner integral asks for $\int \sqrt{1+x^3} \, dx$, which has no elementary antiderivative. No substitution saves it, because there is no x^2 outside the root to make $u = 1+x^3$ work. Switching may bring one in.

Step 1, inequalities. $0 \leq y \leq 1$ and $\sqrt{y} \leq x \leq 1$.

Step 2, sketch. The curve $x = \sqrt{y}$ is the same curve as $y = x^2$ for $x \geq 0$. The line $x = 1$ is the right wall. The strip at height y starts on the parabola and ends at $x = 1$, so the region lies to the right of the parabola and under it: the slate region in Figure 2. Its corners are $(0, 0)$, where $y = 0$ meets the parabola, $(1, 0)$ and $(1, 1)$, where $x = 1$ meets $y = 0$ and the parabola.

Steps 3 and 4, vertical strips. A vertical strip at x enters at the bottom, $y = 0$, and leaves through the parabola, $y = x^2$. The strips run from $x = 0$ to $x = 1$. No curve changes partway, so there is one integral:

$$I = \int_0^1 \int_0^{x^2} \sqrt{1+x^3} dy dx.$$

Inner integral. $\sqrt{1+x^3}$ does not contain y , so it is a constant for the dy integral:

$$\int_0^{x^2} \sqrt{1+x^3} dy = \sqrt{1+x^3} [y]_0^{x^2} = x^2 \sqrt{1+x^3}.$$

The x^2 we needed has appeared. *Outer integral.* Let $u = 1+x^3$, so $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$. Change the limits too: $x = 0$ gives $u = 1$, $x = 1$ gives $u = 2$.

$$I = \int_0^1 x^2 \sqrt{1+x^3} dx = \frac{1}{3} \int_1^2 u^{1/2} du = \frac{1}{3} \cdot \frac{2}{3} [u^{3/2}]_1^2 = \frac{2}{9} (2^{3/2} - 1) = \frac{2}{9} (2\sqrt{2} - 1) \approx 0.4063.$$

WORKED EXAMPLE 2 · TWO PIECES INTO ONE · WRITTEN FOR THIS PACK

Write $J = \int_0^1 \int_0^x xy dy dx + \int_1^2 \int_0^{2-x} xy dy dx$ as a single integral, and evaluate it.

WORKED IN FULL · NO STEP SKIPPED

Inequalities and sketch. The first piece is $0 \leq x \leq 1, 0 \leq y \leq x$: the triangle under the line $y = x$. The second is $1 \leq x \leq 2, 0 \leq y \leq 2 - x$: the triangle under $y = 2 - x$. They share the edge $x = 1$, and together they are one triangle with corners $(0, 0)$, $(2, 0)$ and $(1, 1)$, where the two lines meet. It took two integrals because a vertical strip's top changes from $y = x$ to $y = 2 - x$ at $x = 1$.

Horizontal strips. At height y a strip enters through $y = x$, so $x = y$, and leaves through $y = 2 - x$, so $x = 2 - y$. The heights run from the base $y = 0$ to the apex $y = 1$. Nothing changes along the way, so one integral covers it:

$$J = \int_0^1 \int_y^{2-y} xy dx dy.$$

$$\text{Inner. } \int_y^{2-y} xy dx = y \left[\frac{x^2}{2} \right]_y^{2-y} = \frac{y}{2} ((2-y)^2 - y^2) = \frac{y}{2} (4 - 4y) = 2y - 2y^2.$$

$$\text{Outer. } J = \int_0^1 (2y - 2y^2) dy = \left[y^2 - \frac{2y^3}{3} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

Check. The two original pieces are $\frac{1}{8}$ and $\frac{5}{24}$, and $\frac{3}{24} + \frac{5}{24} = \frac{1}{3}$.

The usual traps

- Swapping the limits without the sketch: $\int_0^1 \int_{\sqrt{y}}^1 \dots dx dy$ is not $\int_0^1 \int_{\sqrt{x}}^1 \dots dy dx$. Always redraw.
- Solving for the wrong root. $y = x^2$ gives $x = \sqrt{y}$ only for $x \geq 0$; on the left half it is $x = -\sqrt{y}$.
- Outer limits with a variable in them. After the switch the outer limits are numbers, always.
- Missing a split. If the new strip crosses a corner where the boundary changes, you need two integrals.

SELF-CHECK · ANSWER ON PAGE 22

Evaluate $\int_0^1 \int_x^1 e^{y^2} dy dx$. (Sketch the triangle, switch, and the y you need appears.)

2 Polar coordinates

When the region is round, describe it by distance from the origin and angle, and pay for the change with one factor of r .

The substitution is $x = r \cos \theta$, $y = r \sin \theta$, so $x^2 + y^2 = r^2$, and the area element is

$$dA = r dr d\theta. \quad (1)$$

The r is the Jacobian $|\partial(x, y)/\partial(r, \theta)|$ (§4). A polar box is wider the farther it sits from the origin, and the r accounts for that.

Recipe

1. Sketch the region. Mark the origin.
2. θ limits: the angles of the two rays that fence the region in. A line $y = mx$ through the origin is the ray $\tan \theta = m$, and you pick the angle in the quadrant you drew.
3. r limits: stand at the origin, walk out along one ray, and note where you enter and leave the region. A circle $x^2 + y^2 = a^2$ is $r = a$.
4. Rewrite the integrand with $x = r \cos \theta$, $y = r \sin \theta$, and multiply by r .

5. Integrate r first, then θ . When the limits are all constants and the integrand is a product, the double integral is a product of two single ones.

WORKED EXAMPLE 3 · POLAR, WITH THE ANGLE READ FROM A LINE · WRITTEN FOR THIS PACK

Evaluate $\iint_D xy \, dA$, where D is the part of the ring $1 \leq x^2 + y^2 \leq 4$ with $0 \leq y \leq \sqrt{3}x$.

WORKED IN FULL · NO STEP SKIPPED

θ . $y \geq 0$ is the upper half-plane, so θ starts at the positive x -axis, $\theta = 0$. The line $y = \sqrt{3}x$ has slope $\sqrt{3}$, so $\tan \theta = \sqrt{3}$ and $\theta = \pi/3$ in the first quadrant. $y \leq \sqrt{3}x$ is the side below that line, so $0 \leq \theta \leq \pi/3$.

r . Along any of those rays you enter at the circle $r = 1$ and leave at $r = 2$. So $1 \leq r \leq 2$.

Integrand. $xy = (r \cos \theta)(r \sin \theta) = r^2 \cos \theta \sin \theta$. With the extra r from (1), it is $r^3 \cos \theta \sin \theta$.

$$\iint_D xy \, dA = \int_0^{\pi/3} \int_1^2 r^3 \cos \theta \sin \theta \, dr \, d\theta = \left(\int_1^2 r^3 \, dr \right) \left(\int_0^{\pi/3} \sin \theta \cos \theta \, d\theta \right).$$

The two factors. $\int_1^2 r^3 \, dr = \left[\frac{r^4}{4} \right]_1^2 = \frac{16-1}{4} = \frac{15}{4}$. For the angle, let $w = \sin \theta$, $dw = \cos \theta \, d\theta$:
 $\int_0^{\pi/3} \sin \theta \cos \theta \, d\theta = \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/3} = \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{3}{8}$.

$$\iint_D xy \, dA = \frac{15}{4} \cdot \frac{3}{8} = \frac{45}{32}.$$

A circle that is not centred at the origin. The disk $x^2 + y^2 \leq 2y$ is the circle of radius 1 centred at $(0, 1)$. Substituting gives $r^2 \leq 2r \sin \theta$, so $0 \leq r \leq 2 \sin \theta$, and $r \geq 0$ forces $\sin \theta \geq 0$, so $0 \leq \theta \leq \pi$. Its area is $\int_0^\pi \int_0^{2 \sin \theta} r \, dr \, d\theta = \int_0^\pi 2 \sin^2 \theta \, d\theta = \pi$, as it should be for a unit disk.

The usual traps

- Forgetting the r . Without it every polar answer is wrong. The quiz may ask exactly this as a true/false.
- Reading θ off the wrong quadrant: $y = -x$ is $\theta = 3\pi/4$ on the left, $-\pi/4$ on the right.
- $(x^2 + y^2)^{3/2}$ is r^3 , and with dA it becomes $r^4 \, dr \, d\theta$.
- A full disk is $0 \leq \theta \leq 2\pi$; the upper half is $0 \leq \theta \leq \pi$.

SELF-CHECK · ANSWER ON PAGE 22

Evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy \, dx$ by switching to polar.

3 Triple integrals: cylindrical and spherical

Read the limits from the inside out: the innermost variable along a line or a ray, the outer ones from the shadow. Then multiply by r or by $\rho^2 \sin \phi$.

Cylindrical is polar in the xy -plane with z left alone: $x = r \cos \theta$, $y = r \sin \theta$, $z = z$. *Spherical* measures the distance ρ from the origin, the angle ϕ down from the positive z -axis ($0 \leq \phi \leq \pi$), and the same θ as polar: $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, and the cylindrical radius is $r = \rho \sin \phi$. The volume elements are

$$dV = r \, dz \, dr \, d\theta \quad \text{and} \quad dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \quad (2)$$

Most of the work is translating surfaces. The table does that for the ones the quiz uses.

SURFACE	CARTESIAN	CYLINDRICAL	SPHERICAL
Sphere, radius a	$x^2 + y^2 + z^2 = a^2$	$r^2 + z^2 = a^2$	$\rho = a$
Cylinder, radius a	$x^2 + y^2 = a^2$	$r = a$	$\rho \sin \phi = a$, so $\rho = a / \sin \phi$
Plane, height c	$z = c$	$z = c$	$\rho \cos \phi = c$, so $\rho = c / \cos \phi$
Cone	$z = k\sqrt{x^2 + y^2}$	$z = kr$	$\phi = \phi_0$ with $\tan \phi_0 = 1/k$
Paraboloid	$z = x^2 + y^2$	$z = r^2$	messy: use cylindrical
Half-plane through the z -axis	$y = x, x > 0$	$\theta = \pi/4$	$\theta = \pi/4$

The cone row is where marks go. On the cone $z = kr$, $\tan \phi = r/z = 1/k$. So $z = \sqrt{x^2 + y^2}$ is $\phi = \pi/4$, and $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$ (PS2 Q9) has $\tan \phi = \sqrt{3}$, so $\phi = \pi/3$.

Recipe: cylindrical

1. Find the floor and the ceiling: the surface the solid sits on and the one that caps it. Write each as $z =$ a function of r . Those are the z limits.
2. Find the shadow on the xy -plane. Its edge is where the floor meets the ceiling (set them equal and solve for r), or a cylinder the question names.
3. Read r and θ off the shadow as in polar (§2).
4. Multiply the integrand by r . Integrate z , then r , then θ .

Recipe: spherical

1. θ : how far round the z -axis the solid goes. A whole solid of revolution is 0 to 2π .
2. ϕ : start on the positive z -axis ($\phi = 0$) and tip down. Where does the solid stop? At a cone $\phi = \phi_0$, or at the xy -plane ($\phi = \pi/2$), or at the negative z -axis ($\phi = \pi$).
3. ρ : for a fixed ϕ , walk out from the origin along that ray. Where you leave is the upper limit, written as $\rho =$ a function of ϕ (table above). Where you enter is the lower limit, usually 0.
4. If the surface you leave through changes as ϕ changes, split at the ϕ where the two surfaces meet. One integral per piece.
5. Multiply the integrand by $\rho^2 \sin \phi$. Integrate ρ , then ϕ , then θ .

WORKED EXAMPLE 4 · CYLINDRICAL: FLOOR AND CEILING · WRITTEN FOR THIS PACK

Find the volume of the solid between the paraboloids $z = x^2 + y^2$ (below) and $z = 2 - x^2 - y^2$ (above).

WORKED IN FULL · NO STEP SKIPPED

Floor and ceiling. In cylindrical, $z = r^2$ below and $z = 2 - r^2$ above, so $r^2 \leq z \leq 2 - r^2$.

Shadow. The two meet where $r^2 = 2 - r^2$, so $r^2 = 1$ and $r = 1$. Inside that circle the ceiling is above the floor (check $r = 0$: $0 < 2$). The shadow is the disk $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r \, dz \, dr \, d\theta.$$

Inside out. $\int_{r^2}^{2-r^2} r \, dz = r[(2 - r^2) - r^2] = 2r - 2r^3$. Then $\int_0^1 (2r - 2r^3) \, dr = [r^2 - \frac{r^4}{2}]_0^1 = \frac{1}{2}$. Then $\int_0^{2\pi} \frac{1}{2} \, d\theta = \pi$.

$$V = \pi.$$

WORKED EXAMPLE 5 · SPHERICAL: ϕ FROM A CONE · WRITTEN FOR THIS PACK

Evaluate $\iiint_E z \, dV$, where E is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

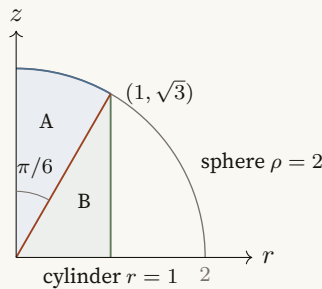
WORKED IN FULL · NO STEP SKIPPED

θ . The solid goes all the way round: $0 \leq \theta \leq 2\pi$. ϕ . From the z -axis down to the cone. The cone $z = r$ has $\tan \phi = r/z = 1$, so $\phi = \pi/4$. Thus $0 \leq \phi \leq \pi/4$. ρ . Every ray in that range leaves through the sphere $\rho = 2$: $0 \leq \rho \leq 2$. *Integrand.* $z = \rho \cos \phi$; times $\rho^2 \sin \phi$ it is $\rho^3 \cos \phi \sin \phi$.

$$\iiint_E z \, dV = \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \cos \phi \sin \phi \, d\phi \cdot \int_0^2 \rho^3 \, d\rho = 2\pi \cdot \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/4} \cdot \left[\frac{\rho^4}{4} \right]_0^2 = 2\pi \cdot \frac{1}{4} \cdot 4 = 2\pi.$$

WORKED EXAMPLE 6 · SPHERICAL LIMITS THAT SPLIT · WRITTEN FOR THIS PACK

Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 4$, inside the cylinder $x^2 + y^2 = 1$, and above the xy -plane.

FIGURE 3 the slice at fixed θ 

A vertical half-plane through the z -axis, with r across and z up. The solid is everything inside the arc $\rho = 2$ and left of the line $r = 1$. A ray with $\phi < \pi/6$ (slate) leaves through the sphere; a ray with $\phi > \pi/6$ (sage) leaves through the cylinder. The rust ray is the split, through the corner $(1, \sqrt{3})$.

WORKED IN FULL · NO STEP SKIPPED

θ and ϕ . All the way round, $0 \leq \theta \leq 2\pi$. Above the xy -plane, $0 \leq \phi \leq \pi/2$.

ρ , and why it splits. Walk out along a ray. Near the z -axis you hit the sphere first, at $\rho = 2$. Near the xy -plane you hit the cylinder first, at $\rho \sin \phi = 1$, so $\rho = 1/\sin \phi$. The switch happens on

the ray through the corner where sphere and cylinder meet: $r = 1$ and $r^2 + z^2 = 4$ give $z = \sqrt{3}$, and $\tan \phi = r/z = 1/\sqrt{3}$, so $\phi = \pi/6$. Two pieces:

$$V = \underbrace{\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}_A + \underbrace{\int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{1/\sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}_B.$$

Piece A. $\int_0^2 \rho^2 \, d\rho = \frac{8}{3}$. $\int_0^{\pi/6} \sin \phi \, d\phi = 1 - \cos \frac{\pi}{6} = 1 - \frac{\sqrt{3}}{2}$. So $A = 2\pi \cdot \frac{8}{3} \left(1 - \frac{\sqrt{3}}{2}\right) = \frac{16\pi}{3} - \frac{8\sqrt{3}\pi}{3}$.

Piece B. $\int_0^{1/\sin \phi} \rho^2 \, d\rho = \frac{1}{3\sin^3 \phi}$. Times $\sin \phi$ this is $\frac{1}{3} \csc^2 \phi$, and $\int \csc^2 \phi \, d\phi = -\cot \phi$. So $\int_{\pi/6}^{\pi/2} \frac{1}{3} \csc^2 \phi \, d\phi = \frac{1}{3}(-\cot \frac{\pi}{2} + \cot \frac{\pi}{6}) = \frac{1}{3}(0 + \sqrt{3}) = \frac{\sqrt{3}}{3}$, and $B = 2\pi \cdot \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}\pi}{3}$.

$$V = \frac{16\pi}{3} - \frac{8\sqrt{3}\pi}{3} + \frac{2\sqrt{3}\pi}{3} = \frac{16\pi}{3} - 2\sqrt{3}\pi = 2\pi \left(\frac{8}{3} - \sqrt{3}\right) \approx 5.872.$$

Check in cylindrical, where it does not split: floor $z = 0$, ceiling $z = \sqrt{4 - r^2}$, shadow $r \leq 1$.
 $V = 2\pi \int_0^1 r \sqrt{4 - r^2} \, dr = 2\pi \left[-\frac{1}{3}(4 - r^2)^{3/2}\right]_0^1 = 2\pi \cdot \frac{8 - 3\sqrt{3}}{3}$, the same number.

The lesson of the last line: when one system splits and another does not, the one that does not is usually the better choice. PS2 Q10 is the same idea with a plane, ρ running from $1/\cos \phi$ to 2.

The usual traps

- Leaving out r or $\rho^2 \sin \phi$. Both come from (2) every time.
- Running ϕ to 2π . It stops at π ; θ is the one that goes round.
- Measuring ϕ up from the xy -plane. It is measured down from the z -axis, so $\tan \phi = r/z$, not z/r .
- Reading the cone backwards: $z = \sqrt{3}r$ is $\phi = \pi/6$, $z = r/\sqrt{3}$ is $\phi = \pi/3$.
- Writing an upper limit for ρ as a constant when the ray leaves through a plane or a cylinder. It is $c/\cos \phi$ or $a/\sin \phi$.

SELF-CHECK · ANSWER ON PAGE 22

(a) The mass of the solid ball $\rho \leq 1$ whose density is its distance from the origin. (b) $\iiint z \, dV$ over the solid under $z = 4 - x^2 - y^2$ and above $z = 0$. (c) The volume of the tetrahedron cut from the first octant by $x + 2y + 2z = 4$ (PS2 Q7, in Cartesian: floor $z = 0$, ceiling the plane, shadow a triangle).

4 Change of variables with the Jacobian

Choose u and v so the region becomes a rectangle; the price is the absolute value of one determinant.

For a map from the uv -plane to the xy -plane,

$$\iint_{D_{xy}} f(x, y) dA = \iint_{D_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv, \quad \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}. \quad (3)$$

The determinant says how much area a small $du dv$ box gains or loses when it is mapped into the xy -plane. Polar is the case $x = r \cos \theta$, $y = r \sin \theta$, whose determinant is r .

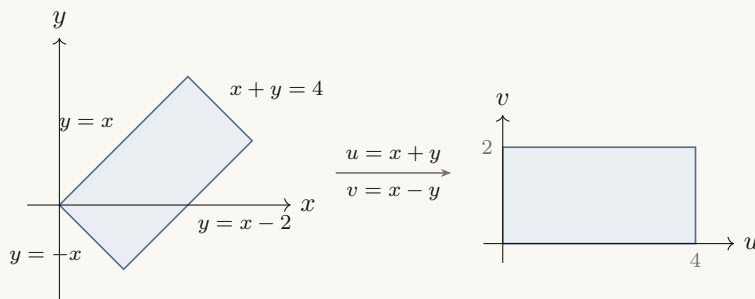
Recipe

1. Look at the boundary curves. If they come in two pairs, each of the form “an expression equals a constant”, set u to one expression and v to the other.
2. The new region is the rectangle between those constants.
3. Compute the easy determinant, $\partial(u, v)/\partial(x, y)$, from u and v as given. Then flip it:

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\partial(u, v)/\partial(x, y)}.$$

4. Take its absolute value. If it still contains x or y , rewrite it in u and v .
5. Rewrite the integrand in u and v , and integrate over the rectangle.

FIGURE 4 a tilted square becomes a rectangle



Left, the slate region in the xy -plane, bounded by four lines. Right, the same region in the uv -plane. The labels on the sides say which line becomes which edge.

WORKED EXAMPLE 7 · JACOBIAN: LINES IN TWO DIRECTIONS · WRITTEN FOR THIS PACK

Evaluate $\iint_D (x^2 - y^2) dA$, where D is bounded by $y = x$, $y = x - 2$, $y = -x$ and $y = 4 - x$.

WORKED IN FULL · NO STEP SKIPPED

Steps 1 and 2. Rewrite the four lines as $x - y = 0$, $x - y = 2$, $x + y = 0$, $x + y = 4$. Set $u = x + y$ and $v = x - y$. The region becomes $0 \leq u \leq 4$, $0 \leq v \leq 2$ (Figure 4).

Step 3. $u_x = 1$, $u_y = 1$, $v_x = 1$, $v_y = -1$, so

$$\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (1)(-1) - (1)(1) = -2, \quad \frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}.$$

Step 4. $|\frac{1}{-2}| = \frac{1}{2}$, so $dA = \frac{1}{2} du dv$. (Solving instead, $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$, gives the determinant $\frac{1}{2} \cdot (-\frac{1}{2}) - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}$ directly.)

Step 5. $x^2 - y^2 = (x + y)(x - y) = uv$.

$$\iint_D (x^2 - y^2) dA = \int_0^2 \int_0^4 uv \cdot \frac{1}{2} du dv = \frac{1}{2} \left[\frac{u^2}{2} \right]_0^4 \left[\frac{v^2}{2} \right]_0^2 = \frac{1}{2} \cdot 8 \cdot 2 = 8.$$

The usual traps

- Using $\partial(u, v)/\partial(x, y)$ where (3) wants its reciprocal. The one in the integral always has the old variables on top.
- Dropping the absolute value. A negative Jacobian only means the map flips orientation; area is never negative.
- Leaving x or y in the Jacobian. In the self-check below it comes out as $2y/x$, which must become $2v$.

SELF-CHECK · ANSWER ON PAGE 22

Find the area of the region in the first quadrant between the hyperbolas $xy = 1$ and $xy = 3$ and the lines $y = x$ and $y = 4x$. Use $u = xy$, $v = y/x$.

5 Vector fields and field lines

A field line is a curve that $\mathbf{F}$ is tangent to everywhere, so its slope is F_2/F_1 .

A vector field $\mathbf{F}(x, y) = (F_1, F_2)$ puts an arrow at every point. A field line $\mathbf{g}(t)$ follows the arrows: $\mathbf{g}'(t) = \mathbf{F}(\mathbf{g}(t))$. In components, $dx/dt = F_1$ and $dy/dt = F_2$, and dividing the two removes t .

Recipe

1. Write $\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)}$.
2. Separate the variables and integrate both sides. Keep the constant: each value of it is one field line.
3. Check the lines where $F_1 = 0$ or $F_2 = 0$ separately. An axis is often a field line too.
4. Put arrows on: evaluate $\mathbf{F}$ at two or three easy points and draw which way it points.
5. A point that lines leave in every direction is a source; one they all enter is a sink.

WORKED EXAMPLE 8 · FIELD LINES · WRITTEN FOR THIS PACK

Find and sketch the field lines of $\mathbf{F}(x, y) = (x, 2y)$.

WORKED IN FULL · NO STEP SKIPPED

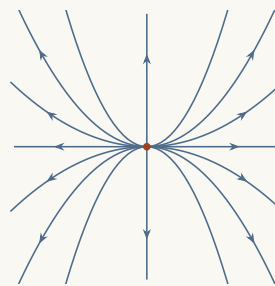
Step 1. $\frac{dy}{dx} = \frac{2y}{x}$.

Step 2. $\frac{dy}{y} = \frac{2 dx}{x}$, so $\ln|y| = 2 \ln|x| + c$, and exponentiating, $y = Cx^2$. The field lines are parabolas through the origin.

Step 3. $x = 0$: then $\mathbf{F} = (0, 2y)$ points straight along the y -axis, so the y -axis is a field line (C “infinite”). $C = 0$ gives the x -axis.

Step 4. At $(1, 0)$, $\mathbf{F} = (1, 0)$ points right. At $(0, 1)$, $\mathbf{F} = (0, 2)$ points up. At $(-1, -1)$, $\mathbf{F} = (-1, -2)$ points down and left. Every arrow points away from the origin.

Step 5. The origin is a source.



Field lines $y = Cx^2$ of $\mathbf{F} = (x, 2y)$, with both axes. Arrows point out of the source at the origin.

The usual traps

- No arrows. The curves alone are half the answer; the direction is the other half.
- Losing a line by dividing by zero. Check $F_1 = 0$ and $F_2 = 0$ before you divide.
- The field lines of a gradient $\nabla\phi$ cross the level curves of ϕ at right angles (PS3 Q12). That is a likely true/false.

SELF-CHECK · ANSWER ON PAGE 22

Find the field lines of $\mathbf{F}(x, y) = (y, x)$ and the direction of the arrow at $(1, 0)$.

6 Scalar line integrals: $\int_C f ds$

ds is length, so $ds = \|\mathbf{g}'(t)\| dt$: the speed of the parametrisation times dt .

With C traced by $\mathbf{x} = \mathbf{g}(t)$ for $a \leq t \leq b$,

$$\int_C f ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt. \quad (4)$$

$f(\mathbf{g}(t))$ is the density at the point you are at, and $\|\mathbf{g}'(t)\| dt$ is how much length you cover in dt . With $f = 1$ the integral is the arc length.

Recipe

1. Parametrise C . Three shapes cover most questions:
 - the segment from A to B : $\mathbf{g}(t) = A + t(B - A)$, $0 \leq t \leq 1$;
 - the circle of radius R : $\mathbf{g}(t) = (R \cos t, R \sin t)$, a quarter of it is $0 \leq t \leq \pi/2$;
 - the graph $y = h(x)$: $\mathbf{g}(t) = (t, h(t))$, with t running over the x -values.
2. Differentiate each component: $\mathbf{g}'(t)$.
3. The norm, in four lines. Square each component of $\mathbf{g}'(t)$. Add the squares. Simplify: expand, cancel cross terms, use $\sin^2 t + \cos^2 t = 1$. Take the square root.
4. Substitute the components of $\mathbf{g}(t)$ into f to get $f(\mathbf{g}(t))$, a function of t only.
5. Integrate $f(\mathbf{g}(t)) \|\mathbf{g}'(t)\|$ from the smaller t to the larger one.

WORKED EXAMPLE 9 • SCALAR LINE INTEGRAL ON A CIRCLE • WRITTEN FOR THIS PACK

Evaluate $\int_C xy^2 \, ds$, where C is the quarter of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$.

WORKED IN FULL • NO STEP SKIPPED

Step 1. Radius 2: $\mathbf{g}(t) = (2 \cos t, 2 \sin t)$. At $t = 0$ it is at $(2, 0)$ and at $t = \pi/2$ at $(0, 2)$, so $0 \leq t \leq \pi/2$.

Step 2. $\mathbf{g}'(t) = (-2 \sin t, 2 \cos t)$.

Step 3, the norm.

$$\|\mathbf{g}'(t)\| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} = \sqrt{4 \sin^2 t + 4 \cos^2 t} = \sqrt{4(\sin^2 t + \cos^2 t)} = \sqrt{4} = 2.$$

The speed is the radius, not 1.

Step 4. $x = 2 \cos t$ and $y = 2 \sin t$, so $f(\mathbf{g}(t)) = (2 \cos t)(2 \sin t)^2 = 8 \cos t \sin^2 t$.

Step 5.

$$\int_C xy^2 \, ds = \int_0^{\pi/2} 8 \cos t \sin^2 t \cdot 2 \, dt = 16 \int_0^{\pi/2} \sin^2 t \cos t \, dt.$$

Let $w = \sin t$, $dw = \cos t \, dt$; $t = 0$ gives $w = 0$ and $t = \pi/2$ gives $w = 1$: $16 \int_0^1 w^2 \, dw = 16 \cdot \frac{1}{3} = \frac{16}{3}$.

A segment, for comparison. From $A = (1, 0, 0)$ to $B = (3, 2, 1)$: $\mathbf{g}(t) = (1 + 2t, 2t, t)$, $\mathbf{g}'(t) = (2, 2, 1)$, $\|\mathbf{g}'\| = \sqrt{4 + 4 + 1} = 3$, which is the length of AB . So $\int_C x \, ds = \int_0^1 (1 + 2t) \cdot 3 \, dt = 3[t + t^2]_0^1 = 6$.

When the norm looks hard. In PS3 Q5, $\mathbf{g}'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$. Expand both squares completely: the cross terms $-2t \cos t \sin t$ and $+2t \sin t \cos t$ cancel, $\cos^2 t + \sin^2 t = 1$ and $t^2 \sin^2 t + t^2 \cos^2 t = t^2$, so the sum is $1 + t^2 + 1$ and $\|\mathbf{g}'\| = \sqrt{2 + t^2}$. Writing out every term is what makes it come apart.

The usual traps

- Writing dt for ds , which drops $\|\mathbf{g}'(t)\|$ altogether.
- Taking the speed of a radius- R circle as 1. It is R .
- Direction. $\int_C f \, ds$ does not depend on which way C is traced; integrate from the smaller t to the larger.
- Stopping at $\sqrt{\dots}$. Expand, cancel, use $\sin^2 + \cos^2 = 1$; the root almost always simplifies on a quiz.

SELF-CHECK · ANSWER ON PAGE 22

Evaluate $\int_C (x^2 + y^2 + z^2) ds$ along the helix $\mathbf{g}(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$.

7 Vector line integrals and work: $\int_C \mathbf{F} \cdot d\mathbf{x}$

Now direction matters: you add up the part of $\mathbf{F}$ along the direction of travel, so the velocity $\mathbf{g}'(t)$ goes in, not its length.

With C traced by $\mathbf{g}(t)$ from $t = a$ (the start) to $t = b$ (the end),

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt. \quad (5)$$

Reverse the direction of travel and $\mathbf{g}'$ reverses, so the answer changes sign.

Recipe

1. Is the path messy, or is the field likely a gradient? Run the test in §8 first; if it passes, stop here and use ϕ .
2. Parametrise C in the direction of travel, so $t = a$ is the start and $t = b$ the end.
3. Differentiate: $\mathbf{g}'(t)$.
4. Substitute: $\mathbf{F}(\mathbf{g}(t))$, every x, y, z in $\mathbf{F}$ replaced by the components of $\mathbf{g}(t)$.
5. Dot product, component by component, to get one function of t . Integrate from a to b .

WORKED EXAMPLE 10 · WORK ALONG TWO PATHS · WRITTEN FOR THIS PACK

Let $\mathbf{F}(x, y) = (y^2, x)$. Find $\int_C \mathbf{F} \cdot d\mathbf{x}$ from $(0, 0)$ to $(1, 1)$ (a) along the parabola $y = x^2$ and (b) along the line $y = x$.

WORKED IN FULL · NO STEP SKIPPED

(a) *Parametrise.* The graph $y = x^2$: $\mathbf{g}(t) = (t, t^2)$, from $t = 0$ at $(0, 0)$ to $t = 1$ at $(1, 1)$.
Differentiate. $\mathbf{g}'(t) = (1, 2t)$. *Substitute.* $F_1 = y^2 = (t^2)^2 = t^4$ and $F_2 = x = t$, so
 $\mathbf{F}(\mathbf{g}(t)) = (t^4, t)$. *Dot.* $(t^4, t) \cdot (1, 2t) = t^4 + 2t^2$.

$$\int_0^1 (t^4 + 2t^2) dt = \frac{1}{5} + \frac{2}{3} = \frac{13}{15}.$$

(b) $\mathbf{g}(t) = (t, t)$, $0 \leq t \leq 1$; $\mathbf{g}'(t) = (1, 1)$; $\mathbf{F}(\mathbf{g}(t)) = (t^2, t)$; dot product $t^2 + t$.

$$\int_0^1 (t^2 + t) dt = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Reading it. Same endpoints, different answers ($\frac{13}{15} \neq \frac{5}{6}$), so this integral depends on the path and $\mathbf{F}$ cannot be a gradient. The test agrees: $\partial F_1 / \partial y = 2y$ but $\partial F_2 / \partial x = 1$. Traced backwards, from $(1, 1)$ to $(0, 0)$ along the parabola, the answer would be $-\frac{13}{15}$.

The usual traps

- Using $\|\mathbf{g}'(t)\|$ here. That belongs to ds ; work uses the vector $\mathbf{g}'(t)$ and a dot product.
- Writing $\mathbf{F}(t)$: substituting t for x in every slot instead of the components of $\mathbf{g}(t)$.
- Parametrising against the direction asked for, which flips the sign.
- Expecting 0 on every closed curve. That holds only for conservative fields; $\mathbf{F} = (y, -x)$ around the unit circle counterclockwise gives -2π .
- A field everywhere perpendicular to the path does no work (PS3 Q8): the dot product is 0 at every t .

SELF-CHECK · ANSWER ON PAGE 22

Find the work done by $\mathbf{F}(x, y, z) = (-y, x, z)$ along the helix $\mathbf{g}(t) = (\cos t, \sin t, t)$ from $t = 0$ to $t = 2\pi$.

8 Conservative fields and the fundamental theorem

If $\mathbf{F} = \nabla\phi$, the line integral is $\phi(\text{end}) - \phi(\text{start})$: no parametrisation, the same on every path, and zero around every closed curve.

The fundamental theorem for line integrals: if $\mathbf{F} = \nabla\phi$ and C runs from A to B , then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \phi(B) - \phi(A). \quad (6)$$

It is the chain rule, $\frac{d}{dt}\phi(\mathbf{g}(t)) = \nabla\phi(\mathbf{g}(t)) \cdot \mathbf{g}'(t)$, integrated from a to b . A field with a potential is called conservative, or a gradient field.

The test. If $\mathbf{F} = \nabla\phi$ then $F_1 = \phi_x$ and $F_2 = \phi_y$, and since $\phi_{xy} = \phi_{yx}$,

$$\text{2D: } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}; \quad \text{3D: } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \quad (7)$$

Failing (7) anywhere proves there is no potential. Passing it guarantees one when $\mathbf{F}$ is smooth on a region with no holes, such as all of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Recipe

1. Test with (7). If a pair differs, stop: parametrise as in §7.
2. Integrate F_1 with respect to x . The “constant” is a function of the other variables: $\phi = \int F_1 dx + h(y)$ in 2D, $+h(y, z)$ in 3D.
3. Differentiate that ϕ with respect to y and set it equal to F_2 . Solve for h_y . It must not contain x ; if it does, you made an error or the test fails.
4. Integrate h_y with respect to y . In 3D the new “constant” is $k(z)$; differentiate ϕ with respect to z , match F_3 , and solve for $k'(z)$.
5. Check: differentiate your ϕ and see every component of $\mathbf{F}$ come back.
6. Evaluate $\phi(\text{end}) - \phi(\text{start})$.

WORKED EXAMPLE 11 · 2D: TEST, POTENTIAL, ENDPOINTS · WRITTEN FOR THIS PACK

Let $\mathbf{F}(x, y) = (2xy + 3, x^2 + 4y^3)$. Find $\int_C \mathbf{F} \cdot d\mathbf{x}$ along any path from $(0, 1)$ to $(2, -1)$.

WORKED IN FULL · NO STEP SKIPPED

Test. $\frac{\partial F_1}{\partial y} = 2x$ and $\frac{\partial F_2}{\partial x} = 2x$. Equal, and $\mathbf{F}$ is a polynomial defined everywhere, so a potential exists and the path does not matter.

Integrate F_1 in x . $\phi = \int (2xy + 3) dx = x^2y + 3x + h(y)$. The y is held fixed, so $\int 2xy dx = x^2y$.

Match F_2 . $\phi_y = x^2 + h'(y)$, which must equal $x^2 + 4y^3$. So $h'(y) = 4y^3$, free of x as it should be, and $h(y) = y^4$ (any constant drops out of the difference).

$$\phi(x, y) = x^2y + 3x + y^4.$$

Check. $\phi_x = 2xy + 3 = F_1$ and $\phi_y = x^2 + 4y^3 = F_2$.

Endpoints. $\phi(2, -1) = 4(-1) + 6 + 1 = 3$ and $\phi(0, 1) = 0 + 0 + 1 = 1$.

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \phi(2, -1) - \phi(0, 1) = 3 - 1 = 2.$$

Along the straight segment $\mathbf{g}(t) = (2t, 1 - 2t)$ the direct integral also gives 2, as it must.

WORKED EXAMPLE 12 · 3D: THREE PAIRS, TWO "CONSTANTS" · WRITTEN FOR THIS PACK

Let $\mathbf{F}(x, y, z) = (y^2 + 2, 2xy + z^3, 3yz^2)$. Find $\int_C \mathbf{F} \cdot d\mathbf{x}$ along any path from $(0, 0, 0)$ to $(1, 2, 1)$.

WORKED IN FULL · NO STEP SKIPPED

Test, all three pairs. $\frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}$. $\frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}$. $\frac{\partial F_2}{\partial z} = 3z^2 = \frac{\partial F_3}{\partial y}$. All equal, defined everywhere: conservative.

x first. $\phi = \int (y^2 + 2) dx = xy^2 + 2x + h(y, z)$.

Match F_2 . $\phi_y = 2xy + h_y$, which must equal $2xy + z^3$, so $h_y = z^3$ and $h = yz^3 + k(z)$. Now $\phi = xy^2 + 2x + yz^3 + k(z)$.

Match F_3 . $\phi_z = 3yz^2 + k'(z)$, which must equal $3yz^2$, so $k'(z) = 0$ and k is a constant.

$$\phi(x, y, z) = xy^2 + 2x + yz^3.$$

Endpoints. $\phi(1, 2, 1) = 4 + 2 + 2 = 8$ and $\phi(0, 0, 0) = 0$, so $\int_C \mathbf{F} \cdot d\mathbf{x} = 8$.

BUT THE TEST PASSED,
SO A LOOP MUST GIVE
ZERO

Only on a region with no holes. $\mathbf{F} = (-y, x)/(x^2 + y^2)$ passes (7) at every point where it is defined, yet around the unit circle counterclockwise $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = \sin^2 t + \cos^2 t = 1$, and the integral is 2π . The field is undefined at the origin, the circle goes round that hole, and the guarantee does not apply. A polynomial field has no such hole.

The usual traps

- Writing $+C$ as a number in step 2. It is a function of the variables you did not integrate over.
- Start minus end. It is $\phi(\text{end}) - \phi(\text{start})$.
- Checking one pair in 3D and stopping. All three must agree.
- Parametrising a path you never needed. "Any curve", a hideous $\mathbf{g}(t)$ (PS3 Q9, Q10) or "path independent?" all mean: test first.

SELF-CHECK · ANSWER ON PAGE 22

Let $\mathbf{F}(x, y) = (e^y, xe^y + 2y)$. Show it is conservative and find $\int_C \mathbf{F} \cdot d\mathbf{x}$ from $(0, 0)$ to $(1, \ln 2)$.
For a 3D one, PS3 Q11 (its answer is in the PS3 solutions).

9 Concept check

Quiz-style questions with one answer each. Decide before you look at the back, and say why in one line.

1. True or false: in polar coordinates $dA = dr d\theta$.
2. True or false: $\mathbf{F} = (2x, 2y)$ is conservative.
3. True or false: $\mathbf{F} = (y, -x)$ is conservative.
4. True or false: reversing the direction of C changes the sign of $\int_C f ds$.
5. True or false: reversing the direction of C changes the sign of $\int_C \mathbf{F} \cdot d\mathbf{x}$.
6. True or false: if $\mathbf{F} = \nabla\phi$ with ϕ smooth everywhere, then $\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$ for every closed curve C .
7. True or false: $\mathbf{F} = (-y, x)/(x^2 + y^2)$ passes the mixed-partials test wherever it is defined, so its integral around the unit circle is 0.
8. True or false: the work done by $\mathbf{F}$ along a curve that is perpendicular to $\mathbf{F}$ at every point is 0.
9. True or false: the field lines of $\nabla\phi$ cross the level curves of ϕ at right angles.
10. In spherical coordinates the cone $z = \sqrt{x^2 + y^2}$ is (a) $\phi = \pi/3$ (b) $\phi = \pi/4$ (c) $\phi = \pi/6$ (d) $\theta = \pi/4$.
11. In spherical coordinates the plane $z = 3$ is (a) $\rho = 3$ (b) $\rho = 3 \sin \phi$ (c) $\rho = 3/\cos \phi$ (d) $\phi = 3$.
12. The best first move on $\int_0^1 \int_x^1 \sin(y^2) dy dx$ is (a) $u = y^2$ in the inner integral (b) switch the order (c) polar (d) integrate by parts.
13. For $\mathbf{g}(t) = (3 \cos t, 3 \sin t, 4t)$, $\|\mathbf{g}'(t)\|$ is (a) 1 (b) 3 (c) 5 (d) 7.
14. The factor that goes with $d\rho d\phi d\theta$ is (a) $\rho \sin \phi$ (b) $\rho^2 \sin \phi$ (c) $\rho^2 \cos \phi$ (d) r .

15. Does $\int_C (y^2 dx + x dy)$ from $(0, 0)$ to $(1, 1)$ depend on the path? (a) yes (b) no (c) only for closed paths (d) cannot tell.
16. $\int_C \mathbf{F} \cdot d\mathbf{x}$ for $\mathbf{F} = (y, x)$ from $(0, 0)$ to $(2, 3)$ along any path is (a) 0 (b) 5 (c) 6 (d) depends on the path.

10 Answers

Each answer was re-derived independently before it went in.

Self-checks

§1 The region is $0 \leq x \leq y \leq 1$; switched, $\int_0^1 \int_0^y e^{y^2} dx dy = \int_0^1 ye^{y^2} dy = \frac{1}{2} [e^{y^2}]_0^1 = \frac{e-1}{2} \approx 0.859$.

§2 The upper half of the unit disk: $\int_0^\pi \int_0^1 e^{r^2} r dr d\theta = \pi \cdot \frac{e-1}{2} \approx 2.699$.

§3 (a) $\int_0^{2\pi} \int_0^\pi \int_0^1 \rho \cdot \rho^2 \sin \phi d\rho d\phi d\theta = 2\pi \cdot 2 \cdot \frac{1}{4} = \pi$. (b) $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} zr dz dr d\theta = \pi \int_0^2 (4-r^2)^2 r dr = \frac{32\pi}{3}$. (c) $\int_0^4 \int_0^{2-x/2} \int_0^{(4-x-2y)/2} dz dy dx = \frac{8}{3}$.

§4 $\partial(u, v)/\partial(x, y) = \det \begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} = \frac{2y}{x} = 2v$, so $dA = \frac{1}{2v} du dv$ on $1 \leq u \leq 3$, $1 \leq v \leq 4$. Area $= \int_1^3 \int_1^4 \frac{1}{2v} dv du = 2 \cdot \frac{1}{2} \ln 4 = \ln 4 \approx 1.386$.

§5 $dy/dx = x/y$, so $y dy = x dx$ and $y^2 - x^2 = C$: hyperbolas, with the lines $y = \pm x$ for $C = 0$. At $(1, 0)$, $\mathbf{F} = (0, 1)$ points up.

§6 $\mathbf{g}' = (-\sin t, \cos t, 1)$, $\|\mathbf{g}'\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, and $x^2 + y^2 + z^2 = 1 + t^2$. $\sqrt{2} \int_0^{2\pi} (1 + t^2) dt = \sqrt{2}(2\pi + \frac{8\pi^3}{3}) \approx 125.8$.

§7 $\mathbf{F}(\mathbf{g}(t)) = (-\sin t, \cos t, t)$, $\mathbf{g}' = (-\sin t, \cos t, 1)$, dot product $1 + t$. $\int_0^{2\pi} (1 + t) dt = 2\pi + 2\pi^2 \approx 26.02$.

§8 $\partial_y e^y = e^y = \partial_x (xe^y + 2y)$. $\phi = xe^y + y^2$, and $\phi(1, \ln 2) - \phi(0, 0) = 2 + (\ln 2)^2 \approx 2.480$.

Concept check

- False: $dA = r dr d\theta$.
- True: $\phi = x^2 + y^2$.

3. False: $\partial_y F_1 = 1$ but $\partial_x F_2 = -1$; around the unit circle it gives -2π .
4. False: ds is length and has no direction.
5. True: $\mathbf{g}'$ reverses.
6. True: start and end are the same point, so $\phi(B) - \phi(A) = 0$.
7. False: it gives 2π . The origin is a hole, so the test guarantees nothing.
8. True: the dot product is 0 at every point.
9. True: a gradient is perpendicular to the level curve through the point.
10. (b): $\tan \phi = r/z = 1$.
11. (c): $z = \rho \cos \phi = 3$.
12. (b): the inner integral has no elementary antiderivative; switched, $\int_0^1 \int_0^y \sin(y^2) dx dy = \int_0^1 y \sin(y^2) dy$.
13. (c): $\sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = \sqrt{25} = 5$.
14. (b): from (2).
15. (a): it is Worked example 10, with $\frac{13}{15}$ and $\frac{5}{6}$ on two paths.
16. (c): $\phi = xy$, and $\phi(2, 3) - \phi(0, 0) = 6$.

Every worked answer and self-check re-derived exactly with sympy on 5 October 2026; the region-based ones (Worked examples 1, 3, 7 and the Jacobian self-check) also checked numerically in the original variables.

Sources

Problem-set questions are cited where used; every other example was written for this pack. No Möbius question is used.

- 1 ECE 206, *Quiz 2 Info*, LEARN announcement, 30 September 2026: scope and format.
- 2 ECE 206, *Problem Set 2* and *Problem Set 2 Solutions*: Q1(b), Q7, Q9, Q10.
- 3 ECE 206, *Problem Set 3* and *Problem Set 3 Solutions*: Q5, Q8, Q9, Q10, Q11, Q12.
- 4 ECE 206, Lectures 4 to 7, and the pre-lecture primers *Vector Calculus 2* and 3: notation for multiple integrals, vector fields and line integrals.
- 5 The ECE 206 lessons *Multiple integrals* and *Line integrals*: <https://lessons.ihsan.cc/ece206/multiple-integrals/>
<https://lessons.ihsan.cc/ece206/line-integrals/>