

Möbius 3 study guide

The seven kinds of question on the third assignment: how to recognise each one, the method, one example of its own worked with nothing skipped, the traps, and a self-check with its answer at the back.

This guide never works an assignment question: every field, curve and number in it was written for it, and the assignment is yours to do. Find your question's kind in the map, read that section's recipe, and follow the worked example with a pencil, doing each line before you read it. When you stall halfway through a computation, the recipe's step number tells you what the next line is.

The map

THE QUESTION ASKS	METHOD IN ONE LINE	WHERE
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$ for a curve given as $\mathbf{g}(t)$ in 3D	Replace x, y, z in $\mathbf{F}$ by the parts of $\mathbf{g}(t)$, dot with $\mathbf{g}'(t)$, integrate in t	\$1 here; crash course \$7
Find a scalar potential ϕ , then the integral along “any curve” from A to B	Test, integrate one component at a time, check by differentiating, then $\phi(B) - \phi(A)$	\$2 here; crash course \$8
For which fields must $\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$ on every closed curve?	The mixed-partial test on each field; equal everywhere means yes	\$3 here; crash course \$8
Which other curves give the same value as $\int_C \mathbf{F} \cdot d\mathbf{x}$?	Test once, then $\phi(\text{end}) - \phi(\text{start})$ for each curve; never parametrise	\$4
A picture of paths and a constant field: is the integral positive, negative or zero?	For $\mathbf{F} = (a, b)$ it is $a \Delta x + b \Delta y$: only the two ends count	\$5
A field line passes through a point; which other points is it on?	Solve $dy/dx = F_2/F_1$, fix the constant with the point, test each point, watch for a second branch	\$6; crash course \$5
A plot of a field: what shape are the equipotentials, and how are they spaced?	They cross the arrows at right angles; the gap between them is $\Delta\phi/\ \mathbf{F}\ $	\$7

Rows two to five are one idea seen four ways. When $\mathbf{F} = \nabla\phi$, the work along a path is $\phi(\text{end}) - \phi(\text{start})$, so the path between the two ends never matters (crash course \$8). Sections 1 and 2 are short because the crash course has the long versions; sections 3 to 7 get most of the room.

Typing answers into Möbius. Enter ϕ without the $+C$, and differentiate it once before you type it, because that check takes thirty seconds and catches most errors. A “Number” box takes an exact value such as $\pi/4-2$, so you do not need to round.

1 Work along a curve given as $\mathbf{g}(t)$

Every x , y and z in $\mathbf{F}$ becomes a function of t , and the dot product with $\mathbf{g}'(t)$ leaves one ordinary integral.

With C traced by $\mathbf{g}(t)$ from $t = a$ to $t = b$,

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt. \quad (1)$$

There is no norm here: $\|\mathbf{g}'(t)\|$ belongs to $\int_C f ds$ (crash course §6).

Recipe: the three-row table

1. Write $\mathbf{g}(t) = (x(t), y(t), z(t))$ and read off what to replace: $x \rightarrow x(t)$, $y \rightarrow y(t)$, $z \rightarrow z(t)$.
2. Differentiate each part to get $\mathbf{g}'(t)$. A part that is a constant has derivative 0.
3. Make a table with one row per component: $F_i(\mathbf{g}(t))$ in one column, the matching part of $\mathbf{g}'(t)$ in the next, and their product in the last.
4. Add the three products and simplify to one polynomial or trig expression in t .
5. Integrate from a to b . If the field passes the test in §2, you can skip all of this and use ϕ instead.

WORKED EXAMPLE 1 • WORK ALONG A 3D CURVE • WRITTEN FOR THIS GUIDE

Let $\mathbf{F}(x, y, z) = (y, xz, x + y)$ and let C be $\mathbf{g}(t) = (t^2, 2t, 1 - t)$ for $0 \leq t \leq 1$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$.

WORKED IN FULL • NO STEP SKIPPED

Step 1, what to replace. $x \rightarrow t^2$, $y \rightarrow 2t$, $z \rightarrow 1 - t$.

Step 2, the velocity. $\mathbf{g}'(t) = (2t, 2, -1)$.

Step 3, the table.

COMPONENT	$F_i(\mathbf{g}(t))$	PART OF $\mathbf{g}'(t)$	PRODUCT
$F_1 = y$	$2t$	$2t$	$4t^2$
$F_2 = xz$	$t^2(1 - t) = t^2 - t^3$	2	$2t^2 - 2t^3$
$F_3 = x + y$	$t^2 + 2t$	-1	$-t^2 - 2t$

Step 4, add. $4t^2 + (2t^2 - 2t^3) + (-t^2 - 2t) = -2t^3 + 5t^2 - 2t$.

Step 5, integrate.

$$\int_0^1 (-2t^3 + 5t^2 - 2t) dt = \left[-\frac{t^4}{2} + \frac{5t^3}{3} - t^2 \right]_0^1 = -\frac{1}{2} + \frac{5}{3} - 1 = \frac{-3 + 10 - 6}{6} = \frac{1}{6}.$$

This field fails the test ($\partial_y F_1 = 1$, $\partial_x F_2 = z$), so parametrising was the only way.

The usual traps

- Writing $\mathbf{F}(t)$ by putting t in every slot. $F_2 = xz$ becomes $x(t)z(t) = t^2(1-t)$, not $t \cdot t$.
- A constant part of $\mathbf{g}$. If $z(t) = 3$ for every t , then $z'(t) = 0$ and the whole third row is 0, but $z = 3$ still goes into F_1 and F_2 wherever they contain z .

SELF-CHECK · ANSWER ON PAGE 16

Let $\mathbf{F}(x, y, z) = (z, x, xy)$ and $\mathbf{g}(t) = (\cos t, \sin t, 2)$ for $0 \leq t \leq \pi/2$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$. (One part of $\mathbf{g}$ is constant.)

2 A potential in 3D, and the integral it gives

Integrate one component, let the “constant” be a function of the other variables, and fix it by matching the next component.

Recipe

1. Test all three pairs: $\partial_y F_1 = \partial_x F_2$, $\partial_z F_1 = \partial_x F_3$, $\partial_z F_2 = \partial_y F_3$.
2. $\phi = \int F_1 dx + h(y, z)$, holding y and z fixed.
3. Differentiate in y , set equal to F_2 , solve for h_y . It must be free of x . Integrate in y : $h = \dots + k(z)$.
4. Differentiate the whole ϕ in z , set equal to F_3 , solve for $k'(z)$, integrate.
5. Check by differentiating ϕ three times. Then $\int_C \mathbf{F} \cdot d\mathbf{x} = \phi(\text{end}) - \phi(\text{start})$.

The quick version. Integrate each component in its own variable: F_1 in x , F_2 in y , F_3 in z . Every term of ϕ shows up in at least one of the three, so write down each distinct term once. This only proposes a ϕ ; step 5 must still confirm it.

WORKED EXAMPLE 2 · POTENTIAL WITH A TWO-VARIABLE LEFTOVER · WRITTEN FOR THIS GUIDE

Let $\mathbf{F}(x, y, z) = (2xy + z^2, x^2 + 3z, 2xz + 3y)$. Find ϕ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$ along any curve from $(1, 0, 2)$ to $(-1, 2, 1)$.

WORKED IN FULL · NO STEP SKIPPED

Step 1, test. $\partial_y F_1 = 2x$ and $\partial_x F_2 = 2x$. $\partial_z F_1 = 2z$ and $\partial_x F_3 = 2z$. $\partial_z F_2 = 3$ and $\partial_y F_3 = 3$. All three pairs agree and $\mathbf{F}$ is a polynomial, so a potential exists.

Step 2. Holding y and z fixed, $\int (2xy + z^2) dx = x^2y + xz^2$. So $\phi = x^2y + xz^2 + h(y, z)$.

Step 3. $\phi_y = x^2 + h_y$. Setting it equal to $F_2 = x^2 + 3z$ gives $h_y = 3z$, which has no x in it. Integrating in y with z fixed, $h = 3yz + k(z)$. Now $\phi = x^2y + xz^2 + 3yz + k(z)$.

Step 4. $\phi_z = 2xz + 3y + k'(z)$. Setting it equal to $F_3 = 2xz + 3y$ gives $k'(z) = 0$, so k is a constant and we drop it.

$$\phi(x, y, z) = x^2y + xz^2 + 3yz.$$

The quick version agrees. $\int F_1 dx = x^2y + xz^2$, $\int F_2 dy = x^2y + 3yz$, $\int F_3 dz = xz^2 + 3yz$. The distinct terms are x^2y , xz^2 and $3yz$.

Step 5, check. $\phi_x = 2xy + z^2$, $\phi_y = x^2 + 3z$, $\phi_z = 2xz + 3y$: all three components come back.

Endpoints. Substitute one coordinate at a time.

$$\phi(-1, 2, 1) = (1)(2) + (-1)(1) + 3(2)(1) = 2 - 1 + 6 = 7, \quad \phi(1, 0, 2) = 0 + (1)(4) + 0 = 4.$$

$$\int_C \mathbf{F} \cdot d\mathbf{x} = 7 - 4 = 3.$$

The usual traps

- Counting a shared term twice. x^2y comes out of both $\int F_1 dx$ and $\int F_2 dy$, but it is one term of ϕ , not $2x^2y$.
- Making h a function of y only in 3D. After the x -integral the leftover can depend on y and z .
- Squaring a negative coordinate wrongly at the end: $(-1)^2 = 1$ but $(-1) \cdot 1 = -1$. Substitute in brackets.

SELF-CHECK · ANSWER ON PAGE 16

Let $\mathbf{F} = (yz + 2x, xz, xy + 2z)$. Find ϕ , and evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$ along any curve from $(0, 1, 1)$ to $(1, 2, 3)$.

3 Must the integral round a closed curve be zero?

“Must be 0 for every closed curve” means “has a potential”. On all of $\mathbb{R}^2$ that is exactly the mixed-partials test.

A closed curve ends where it starts, so a gradient field gives $\phi(A) - \phi(A) = 0$ on every one. A field that fails the test gives a nonzero answer on some closed curve. So the question is the test, field by field:

$$\mathbf{F} = (P, Q) \text{ on all of } \mathbb{R}^2 : \quad \oint_C \mathbf{F} \cdot d\mathbf{x} = 0 \text{ for every closed } C \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ everywhere.} \quad (2)$$

The word “everywhere” carries weight. Two partials that agree only along a line, or at one point, fail.

Recipe

1. Name the parts: P is the first component, Q the second.
2. Strike out what cannot matter. A term of P with no y in it has $\partial_y = 0$, and a term of Q with no x in it has $\partial_x = 0$. Terms like $\sin x$ or x^3 in P , or e^y or y^3 in Q , never decide the answer.
3. Differentiate what is left: $\partial P/\partial y$ and $\partial Q/\partial x$. Use the product rule on anything like xe^{xy} .
4. Compare as functions. Equal for every (x, y) : this field qualifies. Different anywhere: it does not.
5. Tick every field that qualifies. If none does, the answer is “none of the above”.

WORKED EXAMPLE 3 · FOUR FIELDS, WHICH MUST GIVE 0 · WRITTEN FOR THIS GUIDE

For which of these must $\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$ on every closed curve C in $\mathbb{R}^2$?

- (a) $(y^2 + \cos x, 2xy + \frac{1}{1+y^2})$ (b) $(4x^3 - 2y, 2x + y)$
 (c) $(ye^{xy}, xe^{xy} + 1)$ (d) $(3y, 3x + x^2)$

 WORKED IN FULL · NO STEP SKIPPED

(a) Strike $\cos x$ from P and $\frac{1}{1+y^2}$ from Q . Left: $\partial_y(y^2) = 2y$ and $\partial_x(2xy) = 2y$. Equal everywhere, so (a) qualifies. Its potential is $\phi = xy^2 + \sin x + \arctan y$.

(b) Strike $4x^3$ and y . Left: $\partial_y(-2y) = -2$ and $\partial_x(2x) = 2$. They differ, so (b) does not qualify. The sign is the whole question here: the field looks balanced, and it is not.

(c) Nothing to strike. By the product rule, $\partial_y(ye^{xy}) = e^{xy} + y \cdot xe^{xy} = (1 + xy)e^{xy}$, and $\partial_x(xe^{xy} + 1) = e^{xy} + x \cdot ye^{xy} = (1 + xy)e^{xy}$. Equal everywhere, so (c) qualifies, with $\phi = e^{xy} + y$.

(d) $\partial_y(3y) = 3$ and $\partial_x(3x + x^2) = 3 + 2x$. These agree on the line $x = 0$ and nowhere else, so (d) does not qualify.

Answer: (a) and (c).

**BUT I TRIED A CIRCLE
ON (D) AND GOT 0**

Round the unit circle centred at the origin, (d) does give 0, because the mismatch $2x$ is positive on the right half and negative on the left and the two halves cancel. “Must” asks about every closed curve, and one curve that gives a nonzero answer is enough to rule a field out. Round the circle $\mathbf{g}(t) = (1 + \cos t, \sin t)$, $0 \leq t \leq 2\pi$, which sits to the right of the y -axis, (d) gives 2π . Field (b) gives 4π round the unit circle.

The usual traps

- Testing $\partial P/\partial x$ against $\partial Q/\partial y$. The test crosses over: P in y , Q in x .
- Dropping the second term of a product. $\partial_y(ye^{xy})$ has two terms.
- A field with a hole. $(-y, x)/(x^2 + y^2)$ passes the test wherever it is defined and still gives 2π round the unit circle (crash course §8). The equivalence (2) is for fields defined on all of $\mathbb{R}^2$.

 SELF-CHECK · ANSWER ON PAGE 16

Which must give 0 round every closed curve? (i) $(2x + y, x - 3y^2)$ (ii) $(y \cos(xy), x \cos(xy))$
(iii) $(x - y, x + y)$ (iv) $(e^x + y^2, 2xy + 3x)$.

4 Which curves give the same value?

For a gradient field the value is $\phi(\text{end}) - \phi(\text{start})$, so compare that number for each curve and never parametrise.

Recipe

1. Test the field (§3 in 2D, §2 in 3D). If it fails, this shortcut is not available.
2. Find ϕ .
3. For the reference curve, plug its first and last t into $\mathbf{g}(t)$ to get its start and end points. Its value is $V = \phi(\text{end}) - \phi(\text{start})$.
4. For each candidate, do the same: start point, end point, value. Write them in a table.
5. Tick the candidates whose value equals V . A closed candidate has value 0. A candidate traced backwards has value $-V$.

Same endpoints is enough for the same value, but it is not needed. Two different pairs of endpoints can give the same difference of ϕ , which is why step 4 compares values and not points.

Endpoints of a trigonometric curve. For $\mathbf{g}(t) = (R \cos t, R \sin t)$, substitute each limit separately: $t = 0$ gives $(R, 0)$, $t = \pi/2$ gives $(0, R)$, $t = \pi$ gives $(-R, 0)$, $t = 3\pi/2$ gives $(0, -R)$. An ellipse $(a \cos t, b \sin t)$ works the same way with a in the first slot and b in the second. A range of length 2π is a closed curve.

WORKED EXAMPLE 4 · FIVE CANDIDATES, NO PARAMETRISING · WRITTEN FOR THIS GUIDE

Let $\mathbf{F}(x, y) = (2x + y^2, 2xy)$, and let C be $\mathbf{g}(t) = (t, t^3)$ for $-1 \leq t \leq 1$. Which of these give the same value as $\int_C \mathbf{F} \cdot d\mathbf{x}$?

- (a) $(t, t), -1 \leq t \leq 1$ (b) $(\sqrt{2} \cos t, \sqrt{2} \sin t), \frac{5\pi}{4} \leq t \leq \frac{9\pi}{4}$ (c) $(\cos t, \sin t), 0 \leq t \leq 2\pi$
 (d) $(-t, -t^3), -1 \leq t \leq 1$ (e) $(t^2, t), 0 \leq t \leq 1$

WORKED IN FULL · NO STEP SKIPPED

Steps 1 and 2. $\partial_y(2x + y^2) = 2y$ and $\partial_x(2xy) = 2y$, so the field is conservative on $\mathbb{R}^2$. Integrating F_1 in x : $\phi = x^2 + xy^2 + h(y)$. Then $\phi_y = 2xy + h'(y) = 2xy$, so $h' = 0$ and $\phi = x^2 + xy^2$.

Step 3, the reference. $t = -1$ gives $(-1, (-1)^3) = (-1, -1)$ and $t = 1$ gives $(1, 1)$. $\phi(1, 1) = 1 + 1 = 2$ and $\phi(-1, -1) = 1 + (-1)(1) = 0$. So $V = 2 - 0 = 2$.

Step 4, the candidates. For (b), $t = \frac{5\pi}{4}$ gives $\sqrt{2}(-\frac{\sqrt{2}}{2}) = -1$ in both slots, and $t = \frac{9\pi}{4}$ is the same angle as $\frac{\pi}{4}$, which gives $(1, 1)$. For (d), $t = -1$ gives $(1, 1)$ and $t = 1$ gives $(-1, -1)$: the reference's ends, swapped.

CURVE	START	END	$\phi(\text{END}) - \phi(\text{START})$	SAME?
(a)	$(-1, -1)$	$(1, 1)$	$2 - 0 = 2$	yes
(b)	$(-1, -1)$	$(1, 1)$	$2 - 0 = 2$	yes
(c)	$(1, 0)$	$(1, 0)$	0, closed	no
(d)	$(1, 1)$	$(-1, -1)$	$0 - 2 = -2$	no
(e)	$(0, 0)$	$(1, 1)$	$2 - 0 = 2$	yes

Answer: (a), (b) and (e). Candidate (e) starts somewhere else, at $(0, 0)$, yet $\phi(0, 0) = \phi(-1, -1) = 0$, so its value is still 2. Comparing endpoints alone would have missed it.

The usual traps

- Parametrising each candidate. If the field passes the test, the shape between the ends is irrelevant: (b) is a long arc below the origin and gives the same 2 as the straight line (a).
- Thinking a closed curve shares the reference's value. It gives 0, which matches only if V happens to be 0.
- Reading only the shape of a curve and not its t -range. $(\cos t, \sin t)$ over $[0, \pi]$ and over $[\pi, 2\pi]$ are different halves with different ends. Substitute the limits every time.
- Missing the reversal. Same two points, swapped order, opposite sign.

SELF-CHECK · ANSWER ON PAGE 16

Let $\mathbf{F} = (y, x + 2y)$ and let C be the segment $(2t, t)$, $0 \leq t \leq 1$. Which give the same value?
 (i) $(2 \sin t, \sin t)$, $0 \leq t \leq \frac{\pi}{2}$ (ii) $(2 \cos t, \sin t)$, $0 \leq t \leq \frac{\pi}{2}$ (iii) $(2 \cos t, \sin t)$, $0 \leq t \leq 2\pi$
 (iv) $(-2t, 3t)$, $0 \leq t \leq 1$ (v) $(2 - 2t, 1 - t)$, $0 \leq t \leq 1$.

5 The sign of the work, read from a picture

A constant field $\mathbf{F} = (a, b)$ is the gradient of $ax + by$, so the work along any path is $a \Delta x + b \Delta y$, and only where the path starts and ends decides its sign.

Why only the ends count. For constant a and b , $\phi = ax + by$ has $\phi_x = a$ and $\phi_y = b$, so $\mathbf{F} = \nabla\phi$. By the fundamental theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \phi(\text{end}) - \phi(\text{start}) = a(x_{\text{end}} - x_{\text{start}}) + b(y_{\text{end}} - y_{\text{start}}) = a \Delta x + b \Delta y. \quad (3)$$

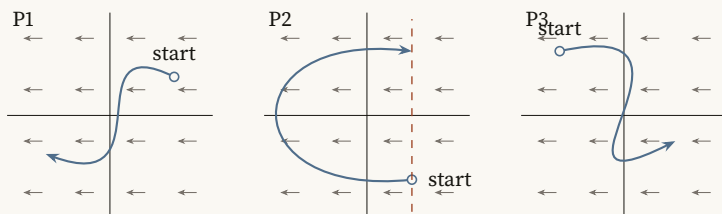
The loops, wiggles and detours in between add nothing, because every stretch that moves with the field is cancelled by a stretch that moves against it. If the field points only along one axis, one of a, b is 0 and the sign comes from one coordinate.

A picture you can carry. Draw the line through the start point at right angles to $\mathbf{F}$. That line is the level set of ϕ through the start. If the path ends on the side $\mathbf{F}$ points towards, the work is positive. If it ends on that line, it is zero. If it ends behind it, it is negative. The same picture works for any gradient field, with the level curve of ϕ through the start in place of the line.

Recipe

1. Find the start and the end. An arrowhead on the curve marks the end. If none is drawn, the start is $\mathbf{g}$ at the smallest t : for $(R \cos t, R \sin t)$ from $t = 0$ that is $(R, 0)$, and the curve then runs counterclockwise.
2. Read Δx and Δy off the axes, end minus start.
3. Compute $a \Delta x + b \Delta y$, or just its sign. Ignore everything between the ends.

FIGURE 1 three paths in the field $\mathbf{F} = (-1, 0)$



Grey arrows are the field, pointing left everywhere. Each slate path starts at the hollow dot and ends at the arrowhead. Only the horizontal distance from start to end matters: P1 ends left of its start, P2 directly above it, P3 to the right.

WORKED EXAMPLE 5 · SIGNS FROM FIGURE 1 · WRITTEN FOR THIS GUIDE

For each path in Figure 1, is $\int_C \mathbf{F} \cdot d\mathbf{x}$ positive, negative or zero, where $\mathbf{F} = (-1, 0)$?

WORKED IN FULL · NO STEP SKIPPED

The formula for this field. $a = -1$ and $b = 0$, so by (3) the work is $-\Delta x$. Its sign is the opposite of the sign of Δx . The rust dashed line in P2 is the level line $x = \text{constant}$ through that start.

P1. It starts at $x = 1$ and ends at $x = -1$, so $\Delta x = -2$ and the work is $-(-2) = 2$: *positive*. The path swings right and left on the way, but those pieces cancel.

P2. It starts at $(0.7, -1)$ and ends at $(0.7, 1)$. $\Delta x = 0$, so the work is *zero*, even though the path travels far to the left and back. On the way out the field helps it, on the way back the field resists it by the same amount. The end lies on the dashed level line through the start.

P3. It starts at $x = -1$ and ends at $x = 0.8$, so $\Delta x = 1.8$ and the work is -1.8 : *negative*. The path moved against the field overall.

A tilted field. For $\mathbf{F} = (2, 1)$ the work is $2\Delta x + \Delta y$. From $(0, 0)$ to $(1, -3)$ that is $2 - 3 = -1$, negative, though the path moved right, the direction of the field's larger component. When neither component is 0, compute the number; the sign of one coordinate is not enough.

The usual traps

- Judging by the length of the path or by how much of it runs with the field. Only the ends count.
- Mixing up start and end. Read the arrowhead, or substitute the smallest t . Reversing a path flips the sign.

SELF-CHECK · ANSWER ON PAGE 16

Let $\mathbf{F} = (1, -1)$. Is the work positive, negative or zero along (i) any path from $(0, 0)$ to $(2, 2)$, (ii) a path that loops twice round the origin on its way from $(1, 3)$ to $(0, 0)$, (iii) any path from $(0, 0)$ to $(-1, 2)$?

6 The field line through a point

Solve $dy/dx = F_2/F_1$, use the given point to fix the constant, then test each candidate point exactly. The line is one connected curve, so a second branch of the same equation is a different line.

A field line is a curve whose tangent is $\mathbf{F}$ at every point (crash course §5). Its equation comes from $dy/dx = F_2/F_1$. That equation has a family of solutions, one per value of the constant, and the point you are given picks out one of them.

Recipe

1. Write $\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)}$. Top is the second component, bottom the first.
2. Separate: everything with y on the left with dy , everything with x on the right with dx . Integrate both sides and add one constant.
3. Substitute the given point and solve for the constant. Write the curve without any constant left.
4. For each candidate point, substitute its x into the curve and compare the y you get with the candidate's y , as exact fractions.
5. Check the candidates that pass: is each on the same piece of the curve as the given point? If the curve has an asymptote or splits into branches, the field line is only the piece through the given point.
6. Check “none of the above” honestly: if every candidate fails step 4 or 5, it is the answer.

WORKED EXAMPLE 6 · A FIELD LINE WITH TWO BRANCHES · WRITTEN FOR THIS GUIDE

A field line of $\mathbf{F}(x, y) = (x, -y)$ passes through $(2, 1)$. Which of these points is it also on: $(1, 2)$, $(3, 1)$, $(4, \frac{1}{2})$, $(-1, -2)$?

WORKED IN FULL · NO STEP SKIPPED

Step 1. $\frac{dy}{dx} = \frac{-y}{x}$.

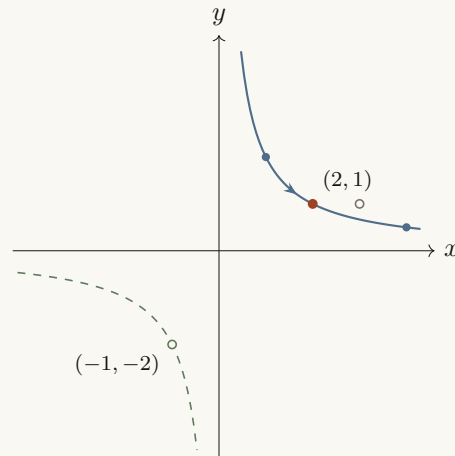
Step 2. Divide by y and multiply by dx : $\frac{dy}{y} = -\frac{dx}{x}$.

Integrate: $\ln|y| = -\ln|x| + c$. Move the log across: $\ln|xy| = c$, so $xy = C$.

Step 3. At $(2, 1)$, $C = 2 \cdot 1 = 2$. The curve is $xy = 2$, that is $y = 2/x$.

Step 4. $(1, 2)$: $2/1 = 2$, yes. $(3, 1)$: $2/3 \neq 1$, no. $(4, \frac{1}{2})$: $2/4 = \frac{1}{2}$, yes. $(-1, -2)$: $2/(-1) = -2$, it satisfies the equation.

Step 5. $y = 2/x$ has two branches, one in each of the first and third quadrants, separated by the axes. To see which the line follows, solve $\mathbf{g}' = \mathbf{F}$ directly: $x' = x$ gives $x = 2e^t$ and $y' = -y$ gives $y = e^{-t}$, starting at $(2, 1)$ when $t = 0$. $x = 2e^t$ is positive for every t , so the line never reaches the third quadrant. Answer: $(1, 2)$ and $(4, \frac{1}{2})$. The point $(-1, -2)$ lies on a different field line with the same equation.



The field line through the rust point (slate, with its direction) is the first-quadrant branch of $xy = 2$. The dashed sage branch satisfies the same equation but is another field line. The hollow grey point is off the curve.

The usual traps

- Flipping the fraction: dy/dx is F_2/F_1 , second over first.
- Testing with decimals. $2/3$ and 0.67 differ; compare fractions.
- Forgetting step 5. An equation like $xy = C$ or $y = 1/(x - k)$ can hold on two separate pieces, and only the one through the given point counts.
- Confusing a field line with a level curve of ϕ . They cross at right angles; the field line follows the arrows.

SELF-CHECK · ANSWER ON PAGE 16

A field line of $\mathbf{F} = (1, y^2)$ passes through $(0, 1)$. Which of $(\frac{1}{2}, 2)$, $(-1, \frac{1}{2})$, $(2, -1)$, $(0, -1)$ is it also on?

7 Equipotentials read from a plot of the field

Equipotentials cross the arrows at right angles, and where the arrows are longer the equipotentials are closer together.

When $\mathbf{F} = \nabla\phi$, an equipotential is a level curve $\phi = \text{constant}$. Two facts let you draw them from a picture of $\mathbf{F}$ alone, without a formula.

Shape. The gradient is perpendicular to the level curve through each point. So at every arrow, the equipotential through its base runs at right angles to it. Arrows that all point the same way give straight equipotentials across them. Arrows that point out from a centre give circles round it.

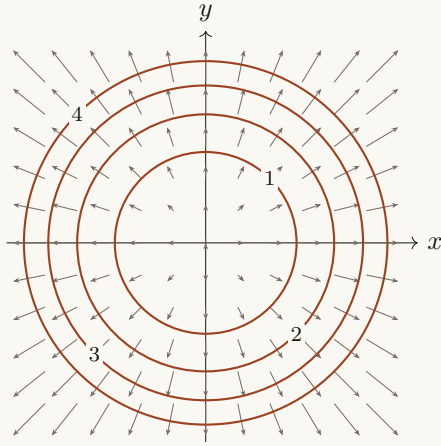
Spacing. Move a short distance Δs along an arrow. The rate of change of ϕ in the direction of $\nabla\phi$ is $\|\nabla\phi\| = \|\mathbf{F}\|$, so ϕ rises by about $\|\mathbf{F}\| \Delta s$. To rise by a fixed step $\Delta\phi$, the step to the next equipotential must be

$$\Delta s \approx \frac{\Delta\phi}{\|\mathbf{F}\|}. \quad (4)$$

In words: long arrows, close lines; short arrows, wide gaps; arrows of one length everywhere, equal gaps. The arrows also tell you which way ϕ increases, because $\mathbf{F}$ points uphill.

Recipe

1. Look at the directions. Sketch two or three short curves that cross the arrows at right angles. Name the family: straight lines (which direction?), circles, something else.
2. Look at the lengths. Walk along the arrows and note whether the arrows get longer, shorter or stay the same.
3. Apply (4): longer means the equipotentials crowd together, shorter means they spread out, the same means equal gaps.
4. If the question restricts to a region, describe only that region. The pattern can be different on the other side of a line.

FIGURE 2 the field $\mathbf{F} = (x, y)$ and its equipotentials

Grey arrows are the field, drawn at a fifth of their true length: they point away from the origin and grow with distance. Rust circles are the equipotentials $\phi = 1, 2, 3, 4$, with $\phi = \frac{1}{2}(x^2 + y^2)$. The circles get closer together as the arrows get longer.

WORKED EXAMPLE 7 · SHAPE AND SPACING FROM A RADIAL PLOT · WRITTEN FOR THIS GUIDE

Figure 2 shows $\mathbf{F} = (x, y)$. Describe its equipotentials, and how they are spaced when drawn at $\phi = 1, 2, 3, \dots$

WORKED IN FULL · NO STEP SKIPPED

Step 1, shape. Every arrow points straight away from the origin. A curve that crosses every such arrow at right angles is a circle centred at the origin. So the equipotentials are circles about the origin.

Confirm with a formula. $\phi = \frac{1}{2}(x^2 + y^2)$ has $\phi_x = x$ and $\phi_y = y$, so $\mathbf{F} = \nabla\phi$. Its level sets $\frac{1}{2}(x^2 + y^2) = k$ are circles of radius $r = \sqrt{2k}$.

Step 2, lengths. $\|\mathbf{F}\| = \sqrt{x^2 + y^2} = r$: arrows grow as you move out.

Step 3, spacing. By (4) the gaps shrink as r grows. The exact radii agree. For $k = 1, 2, 3, 4$ they are $\sqrt{2} \approx 1.414$, 2 , $\sqrt{6} \approx 2.449$ and $\sqrt{8} \approx 2.828$, and the gaps between them are about 0.586 , 0.449 and 0.379 .

Answer: concentric circles about the origin, getting closer together farther out. ϕ increases outward, the way the arrows point.

A second pattern, for contrast. If every arrow points the same way and has the same length, as for $\mathbf{F} = (1, 1)$, the equipotentials are the parallel lines $x + y = c$ at right angles to the arrows, and by (4) they are equally spaced: $\phi = x + y$ rises by the same amount for every equal step along the arrows.

The usual traps

- Drawing equipotentials along the arrows. Those are the field lines (§6); equipotentials cross them.
- Reading spacing backwards. A strong field changes ϕ quickly, so you reach the next value of ϕ in a short step, and the lines are close.
- Describing the whole plane when the question names a region. Answer for the region asked, using the arrows in that region.

SELF-CHECK · ANSWER ON PAGE 16

Describe the equipotentials, and their spacing in equal steps of ϕ , for (i) a plot whose arrows all point down and to the right, along $(1, -1)$, all of one length; (ii) a plot whose arrows all point towards the origin and get shorter farther out, with length $1/r$ at distance r , so $\mathbf{F} = -(x, y)/(x^2 + y^2)$.

8 Answers

Each answer was re-derived exactly with sympy before it went in.

§1 With $x = \cos t$, $y = \sin t$ and $z = 2$, the velocity is $\mathbf{g}' = (-\sin t, \cos t, 0)$ and $\mathbf{F}(\mathbf{g}(t)) = (2, \cos t, \cos t \sin t)$, so the dot product is $-2 \sin t + \cos^2 t$. $\int_0^{\pi/2} (-2 \sin t + \cos^2 t) dt = -2 + \frac{\pi}{4} \approx -1.215$.

§2 All three pairs agree ($z = z$, $y = y$, $x = x$). $\phi = xyz + x^2 + z^2$. $\phi(1, 2, 3) = 6 + 1 + 9 = 16$ and $\phi(0, 1, 1) = 1$, so the integral is 15.

§3 (i) and (ii). (i): $1 = 1$, $\phi = x^2 + xy - y^3$. (ii): both partials are $\cos(xy) - xy \sin(xy)$, $\phi = \sin(xy)$. (iii): $-1 \neq 1$. (iv): $2y \neq 2y + 3$.

- §4 $\phi = xy + y^2$; the reference runs from $(0, 0)$ to $(2, 1)$, so $V = 3$. (i) $(0, 0)$ to $(2, 1)$: 3, yes. (ii) $(2, 0)$ to $(0, 1)$: $1 - 0 = 1$, no. (iii) closed: 0, no. (iv) $(0, 0)$ to $(-2, 3)$: $-6 + 9 = 3$, yes, with different endpoints. (v) reversed: -3 , no. Answer (i) and (iv).
- §5 The work is $\Delta x - \Delta y$. (i) $2 - 2 = 0$: zero. (ii) $(0 - 1) - (0 - 3) = 2$: positive; the loops do not matter. (iii) $-1 - 2 = -3$: negative.
- §6 $dy/dx = y^2$, so $y^{-2} dy = dx$, $-1/y = x + C$; at $(0, 1)$, $C = -1$, and $y = 1/(1 - x)$. $(\frac{1}{2}, 2)$: $1/\frac{1}{2} = 2$, yes. $(-1, \frac{1}{2})$: $1/2$, yes. $(2, -1)$: $1/(1 - 2) = -1$ satisfies the equation, but it lies beyond the asymptote $x = 1$, on the other branch, so no. $(0, -1)$: $1 \neq -1$, no. Answer $(\frac{1}{2}, 2)$ and $(-1, \frac{1}{2})$.
- §7 (i) Lines at right angles to $(1, -1)$, that is $y = x + c$, equally spaced. (ii) Circles about the origin. Here $\phi = -\ln r$, so equal steps in ϕ multiply the radius by the same factor e each time: the gaps widen outward, as (4) predicts for arrows that shrink.

Sources

Every example in this guide was written for it. No Möbius question, field, curve, number or answer is used or reproduced.

- 1 ECE 206, Möbius Assignment 3: the kinds of question only, read to choose this guide's sections.
- 2 ECE 206, *Quiz 2 crash course* (this workspace, October 2026): §5 field lines, §7 work, §8 conservative fields.
- 3 ECE 206, Lectures 6 to 9, and the pre-lecture primers *Vector Calculus 3* and *4*: vector fields, line integrals, the fundamental theorem.
- 4 The ECE 206 lesson *Line integrals*: <https://lessons.ihsan.cc/ece206/line-integrals/>